

**THE DOCTRINE OF RELATIONS IN BERTRAND  
RUSSELL'S  
PRINCIPLES OF MATHEMATICS**

*Michael Pakaluk  
Clark University*

*The question of the nature of relations is of great importance in the early writings of Bertrand Russell, since his disagreements with British Idealism center around relations, and his philosophy of mathematics depends crucially on relations. Yet there is no extended and systematic discussion of relations in early Russell. After examining Russell's definition of a relation, the author systematically and critically examines Russell's views in **Principles of Mathematics** on the following issues: whether a relation exists apart from its terms; the intensional character of relations; difficulties in reflexive relations; and whether simple relations can relate more than two relata.*

The history of early analytic philosophy is a subject most worthy of philosophical study today, especially among philosophers trained in the analytic tradition or influenced by that tradition. Analytic philosophy is often associated with the anti-metaphysical writings of logical positivism and ordinary language philosophy, yet in its formation, especially in the work of Bertrand Russell and O.E. Moore, it was deeply metaphysical in character. Its foundational writings contain discussions which bear a greater resemblance to the later Platonic dialogues, than to the later writings of the analytic school. And the dialectical subtlety of the early period is quite remarkable. To work carefully through a book such as Russell's **Principles of Mathematics** is, I think, to receive some of the training in dialectics which both Plato and Aristotle considered a necessary preliminary to metaphysics.

My topic in the present paper is Russell's views concerning relations in **Principles of Mathematics** (henceforth **Principles**)<sup>1</sup>. Relations are important in early Russell for several reasons. First, the nature of relations was one of the fundamental points of disagreement between Russell and his immediate predecessors, F. H. Bradley and J. McTaggart. In **My Philosophical Development**<sup>2</sup>, Russell, says that the principal focus of his disagreement with his British Idealist teachers was that he argued for pluralism as opposed to monism, and realism as opposed to idealism. Russell believed that both of these issues had to do with relations, and that much of the appeal of monism and idealism derived from mistaken views about relations. It was principally through the doctrine of internal relations, he thought, that the British Idealists argued for monism. The doctrine of internal relations, as Russell understood it, is the view that, for two things to be related, it is necessary that they some how constitute a single whole<sup>3</sup>. If this is the case, and if, as seems plausible, everything bears some relation to anything else, then everything constitutes a single whole. Thus, by arguing against internal relations, Russell took himself to be removing a principal argument for monism. Furthermore, Russell believed that idealism received support from mistaken views about relations. By "idealism" Russell meant the view that everything that exists is dependent upon a perceiving mind for its existence<sup>4</sup>. Russell considered that this

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<sup>1</sup> RUSSELL, Bertrand: *The Principle of Mathematics*, Allen & Unwin, London, 1937, 1903; 1st edn 1093. I shall refer to the work by citing section numbers; and, unless otherwise indicated, any section numbers cited in the paper are for this book.

<sup>2</sup> RUSSELL: *My Philosophical Development*, Allen & Unwin, London, 1959.

<sup>3</sup> This is one of various ways of distinguishing internal from external relations. I put aside the question of whether these various ways of making the distinction are in fact equivalent.

<sup>4</sup> Cp. MOORE, G. E.: "The Refutation of Idealism", *Mind* n. s. 12 (1903), pp. 433-453, reprinted in *Philosophical Studies*, Routledge & Kegan Paul Ltd., London, 1922, pp. 1-30.

view becomes more plausible if one holds that relations do not exist apart from the mind but are rather attributed to reality by the mind. This view, he thought, was held by Leibniz, then Kant, and then embraced by the entire idealist tradition<sup>5</sup>

Second, relations are particularly important in early Russell because of their close connection with logicism, which much of **Principles** is intent on explaining and defending. Logicism is the view that mathematics reduces to logic, i. e. that the truths of mathematics can all be stated using a purely logical vocabulary and can all be derived using only logical axioms and rules of inference. Now much of mathematics has to do with order and with series of terms, and order cannot be accounted for apart from relations. Russell goes so far as to say: *"A careful analysis of mathematical reasoning shows... that types of relations are the true subject-matter discussed, however a bad phraseology may disguise the fact; hence the logic of relations has a more immediate bearing on mathematics than that of classes or propositions, and any theoretically correct and adequate expression of mathematical truths is only possible by its means"*. (§27)

Third, there is a tension in **Principles** between a treatment of logic which aims to be purely extensional, and one which is intensional as well, and this becomes particularly evident in Russell's discussion of relations. Russell shows some ambivalence in **Principles** over whether relations should be understood extensionally or also

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<sup>5</sup> "Thus relations and aggregates [for Leibniz] have only a mental truth; the true proposition is one ascribing a predicate to God and to all others who perceive the relation .... Thus Leibniz is forced, in order to maintain the subject-predicate doctrine, to the Kantian theory that relations, though veritable, are the work of the mind". RUSSELL: *The Philosophy of Leibniz*, Cambridge University Press, Cambridge, 1900, p. 14.

intensionally. He says that relations display "the same rather curious relation of intensional and extensional points of view as do classes" (§98). The intension of a relation seems linked to its 'sense', to the fact that a relation proceeds *from* one term *to* another (§94). He considers that it is this difference of sense which in fact constitutes an asymmetrical, transitive relation - and it is relations of this sort which give rise to order (§§218-9). It would seem, then, that Russell ought to hold that relations considered intensionally are indispensable for providing an account of the foundation of mathematics. Yet in some passages he conjectures that, for the purposes of logic and mathematics, it is possible to dispense with relations in intension and to regard an *n*-adic relation simply as a class of ordered *n*-tuples (cf. §497). By studying the reasons for Russell's ambivalence, it may be possible to acquire a better insight into the role which intension and extension do in fact play in logic.

### 1. The Basic Doctrine of Relations in *Principles*

Russell defines a relation as "*a concept which occurs in a proposition in which there are two terms not occurring as concepts, and in which the interchange of the two terms gives a different proposition*". (§94)<sup>6</sup> "Term" is Russell's most general ontological category; it includes "*whatever may be an object of thought*" and

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<sup>6</sup> Russell gives his basic account of relations in chapter IX of *Principles*, he does so by speaking only about two-term relations: "We mayo .. allow that there are relations having more than two terms; but as these are more complex, it will be well to consider first such as have two terms only". (§94) Hence, following Russell, my explanation of his basic views about relations will also be concerned only about two-term relations. Yet since Russell intended that his account be capable of extension to relations with an indefinite number of terms, it is important to keep in mind the question of whether and in what way his account can be thus generalized and extended.

is used in the sense of "*unit, individual, entity*" (§47). A concept is a term which can be predicated of something; a term "*occurs as a concept*" if it in fact is predicated of something else. In the proposition "*Socrates is human*", *human* occurs as a concept and is a concept, but Socrates neither occurs as nor is a concept. In the proposition "*Socrates has humanity*" (which Russell thinks has the very same constituents as the preceding proposition), *humanity* is a concept but does not occur as a concept: it is a concept, since it is capable of being predicated of another (as indeed it is, in "*Socrates is human*"), but it does not occur as a concept, since it is not predicated of another in this particular proposition. A relation, then, is simply something predicated of two things at once, where the order of the things is important.

Note that this definition does not exclude a relation's being predicated of itself, in cases where the relation occurs in a proposition both as a concept and not as a concept as for example it perhaps does in "*Difference is different from sameness*". Note also that the definition does not require that the things of which the relation is predicated be different either from each other, or from the relation predicated; thus "*Difference is different from difference*" would be a genuine relational proposition, even though false.

There are of course propositions in which one thing is predicated of two things, which we should not, however, consider to be relational propositions, for example: propositions with a compound subject ("*My book and your book are white*") or propositions involving the ascription of a number ("*My book and your book are two*"). Thus it is necessary for Russell to add that "*the interchange of the two terms gives a different proposition*". Russell has in mind propositions such as "*9 is greater than 5*", which actually change in truth-value if the order of things spoken about is reversed: "*9 is greater than 5*" is true; but "*5 is greater than 9*" is false. In con-

trast, "*Your book and my book are two*" or "*Your book and my book are white*" do not change in truth-value, when the order of the subjects is reversed.

However, it is also true in the case of propositions which concern symmetrical relations, that the truth or falsehood of the proposition is not altered if the order of the subjects is changed. For example, "*9 and 32 are equal*" has the same truth value as the proposition in which the subjects are reversed, "*32 and 9 are equal*". So in the stipulation that "*the interchange of the two terms gives a different proposition*" must mean something weaker than *proposition with a different truth value*. Perhaps what Russell intends, then, is that the *meaning* of the proposition is altered if the order of the subjects is altered. But this seems to be too weak and not sufficient to distinguish relational from non-relational propositions. Surely the meaning of "*My book and your book are two*" is different from the meaning of "*Your book and my book are two*", yet that does not imply that either of these statements expresses a relational proposition. Russell's definition, then, fails to distinguish relational from non-relational propositions.

According to Russell, the reason why the order of the subjects is important in a relational proposition is that a proposition has what he calls a "*sense*", which he describes as follows: "*it is characteristic of a relation of two terms that it proceeds, so to speak, **from one to the other**. This is what may be called the **sense** of the relation, and is, as we shall find, the source of order and of series ... The sense of a relation is a fundamental notion, which is not capable of definition*" (§94). Perhaps, then, Russell's definition of a relation should be supplemented by including in it some mention of sense, for example: "*A relation is a concept which occurs in a proposition in which there are two terms not occurring as concepts, and in which the interchange of the two terms gives a proposition with a different*

*sense*". But this is not quite right, since we do not want to say that the proposition has an opposite sense -a claim which is vague, and which might be taken to be about the meaning of the proposition- but rather that the relational concept connecting the two subjects does so in a different manner. Hence, to be more precise, we must say something like: "*A relation is a concept which occurs in a proposition in which there are two terms not occurring as concepts, and which has a sense, in that it proceeds from one of the terms to the other*". By repairing the definition in this way, we make the notion of sense appropriately fundamental to a relation and essential to it.

However, once we define a relational proposition as one involving a concept having a sense, then it is no longer clear that a relation is *said of* both of the subjects in the proposition -or at least it seems that the relation is not said of both of the subjects in the same way, since it is said to be *from* one subject, to the other. Clearly, in the proposition "*A is greater than B*", the concept "*greater than*" is not said of both **A** and **B** in the same way that "*white*" is said of both **A** and **B** in the non-relational proposition, "*A and B are white*". It would even seem justifiable to distinguish the subjects in something like the following manner: a relational proposition is one that is *primarily* about that *from which* the relation proceeds, and *secondarily* about that to which it proceeds. Russell in fact does distinguish the subjects, since he gives distinct names to them: that *from which* the relation proceeds he calls the *referent* of the relation; that to which he calls the *relatum* (§94).

After proposing his definition, Russell makes two assertions about relations which seem to be independent of his definition. He says, first, that every relation **R** has a unique converse relation **R'** such that, if **R** holds from **a** to **b**, then **R'** holds from **b** to **a**, and if **R'** holds from **a** to **b**, **R** holds from **b** to **a** (§94). Hence, as Russell

puts it, a relation "*mutually implies*" its converse. For example, "*greater than*" has a unique converse relation, "*less than*": the proposition that "*9 is greater than 5*" implies the proposition that "*5 is less than 9*"; likewise "*5 is less than 9*" implies "*9 is greater than 5*". Now although these claims about the converse of a relation seem true and unobjectionable, it is worth noting that Russell gives no reason for them, nor does any reason for them appear in his definition of relations. What is it that guarantees that a relation always have a converse? Russell never even raises this question.

The second assertion is this: for any **A**, whatever **A** may be, and for any **B**, whatever **B** may be, there is at least one relation that relates **A** to **B** and which relates no other things. That is, there is a relation which applies uniquely from **A** to **B**. (And assuming that every relation has a converse which it mutually implies, it follows that there is likewise one unique relation that relates **B** to **A** and nothing else and which is mutually implied by the unique relation that relates **A** to **B**). Russell in fact says that: "*The most important of the primitive propositions in this subject [sc. the theory of relations] is that between any two terms there is a relation not holding between any two other terms*" (§28). He says that this is analogous to the principle used in set theory, according to which one postulates, for any thing whatsoever, a unique class which has that thing as its sole member. Russell acknowledges that the principle postulating a unique relation relating any ordered pair is more plausible if relations are understood extensionally -as classes of couples-for then the principle merely asserts that, for every couple, there is a unique class which has that couple as its sole member. But he acknowledges that "*when relations are considered intensionally, it may seem possible to doubt whether the above principle is true at all*". He suggests that it will always be possible to find a unique relation between any two terms simply by finding the logical pro-



duct (the intersection) of all the relations that hold of those two terms: this complex relation, Russell believes, would hold *only of* those two things. But clearly this sort of defense of the principle is unsatisfactory. The principle is proposed as a logical principle concerning relations -as Russell recognizes, for he says "*it may seem possible to doubt*" its truth, as though it is in fact not possible to do so- but this way of defending it turns it into a proposition that holds of everything only accidentally.

Why does Russell want to advance this principle at all? In **Principles**, Russell draws no distinction between logic and metaphysics; he is in fact compelled to consider logic to be simply a very general science of the world, because he thinks that allowing any scope to the activity of the human mind in classifying and organizing knowledge would be to admit idealism. Thus he is unable to adopt any fundamental principle about relations as having application only within a branch of mathematics; rather he is compelled to accept any such principle as a metaphysical principle -even if, as in this case, he recognizes that he cannot defend it as such<sup>7</sup>

## 2. Russell's Rejection of the Classical View of Relations

The classical view of a relation, deriving from Aristotle's **Categories**, is that it is an accident inhering in a subject, which somehow exists *towards* something else. We saw how Russell, in taking the notion of the "*sense*" of a relation as fundamental, and in distin-

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<sup>7</sup> The principle is of course reasonable enough in that branch of mathematics known as the "algebra of relations": the claim that between any two things there is a relation is much like the claim in geometry that between any two points there is a line. There are also technical reasons, related to Russell's logicism, for holding to the principle. Russell wants to define a number as a class of equinumerous classes; but equinumerous classes are simply those whose members can be placed into a one-to-one correspondence; and this one-to-one correspondence in turn consists of arbitrary, postulated relations between the members of one class and those of another.

guishing between the "*referent*" and "*relatum*" of a relation, arrives at a view which resembles the classical view. But Russell considered his view to be quite different from the classical view, as he understood it. Note, however, that in **Principles** Russell never discusses Plato or Aristotle directly, and it is doubtful that he was well acquainted with their views; rather, he takes certain statements about relations found in Leibniz to represent the classical view.

Russell believed that any view which treated a relation as an accident inhering in a single subject is obliged to reduce relations to a qualities or a quantities. His own view, in contrast, is that a relation is not an accident but exists independently between two subjects, connecting the one subject to another.

Russell considers the following passage from Leibniz to be a clear statement of the view that relations are accidents:

*"The ratio or proportion between two fines **L** and **M** may be conceived three several ways; as a ratio of the greater **L** to the lesser **M**; as a ratio of the lesser **M** to the greater **L**; and lastly, as something abstracted from both, that is, as the ratio between **L** and **M**, without considering which is the antecedent, or which is the consequent; which the subject, and which the object ... In the first way of considering them, **L** the greater, in the second **M** the lesser, is the subject of that accident which philosophers call relation. But which of them will be the subject, in the third way of considering them? It cannot be said that both of them, **L** and **M** together, are the subject of such an accident; for if so, we should have an accident in two subjects, with one leg in one, and the other in the other; which is contrary to the notion of accidents. Therefore, we must say that this relation, in this third way of considering it, is*

*indeed out of the subjects; but being neither a substance nor an accident, it must be a mere ideal thing, the consideration of which is nevertheless useful*<sup>8</sup>.

The view that a relation is an "ideal thing" is likened by Russell to the positions of the British Idealists. Russell's own view is that a relation is "out of" its subjects, as is clear from his remarks in **The Philosophy of Leibniz** where he quotes the same text: "*This passage is of capital importance for a comprehension of Leibniz' philosophy. After he has seemed, for a moment, to realize that relation is something distinct from and independent of subject and accident, he thrust aside the awkward discovery, by condemning the third of the above meanings as 'a mere ideal thing'. If he were pushed as to this 'ideal thing', I am afraid he would declare it to be an accident of the mind which contemplates the ratio*" (II, §10).

According to Russell, the view that a relation is an accident requires that a relational proposition, such as "L is greater than M" be understood as the application of an adjective to L:

L is <greater than M>

Russell notes that the adjective is complex and contains reference to M. The adjective is not simply "L is greater": if we were saying simply "L is greater", then we would in no way distinguish L from M, as the original proposition does, since M is also greater (it is no doubt greater than *something*). Furthermore, it must contain some reference to M, otherwise it asserts something short of what was

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<sup>8</sup> LEIBNIZ, G. W.: Die philosophischen Schriften, ed. C. J. Gerhardt, The Prussian Academy of Science, Berlin, 1875-90, Vol. VII, p. 401. Russell's translation.

asserted by the original proposition. But, Russell says, "*and adjective involving reference to M is plainly an adjective which is relative to M, and this is merely a cumbrous way of describing a relation*" (§214). Thus we have not succeeded in understanding "*L is greater than M*" as an accident inhering in **L**, since it has ended up being a relation after all.

This argument is not very convincing. One might concede that a relation involves reference to some other thing, while insisting that a relation nevertheless inheres in only one thing. All that the argument establishes is that a relation inhering in subject must involve reference to some other thing, i.e. must be a relation. The argument does not establish that relation must exist independently of its subjects.

Russell then considers asymmetrical quantitative relations. He wishes to show that the notion of a relational accident inhering only in the referent presupposes the notion of a relation as connecting equally both the referent and the relatum. Consider, he says, a true quantitative proposition having the form "*A is greater than B*". What is it that makes this proposition true? Is it some thing which inheres only in **A** or only in **B**? Presumably, the proposition is true because of the quantities which **A** and **B** have -because **A** has a some quantity, which inheres in **A**, and **B** has another quantity, which inheres in **B**. Suppose that the quantity of **A** is 2 and the quantity of **B** is 1. Then the proposition is true because **A** is 2 and **B** is 1. Yet this does not give a complete account of the original proposition, for that was true because **A** is greater than **B** -that is, because **A** is 2, **B** is 1, and  $2 > 1$ . But if we take "**A** is 2; **B** is 1; and  $2 > 1$ " to be an analysis of "*A is greater than B*", we make use of the relational proposition " $2 > 1$ ", which, it seems, cannot in turn be analyzed into statements about accidents which inhere in single subjects. Russell summarizes by saying: "*This shows that some asymmetrical*

*relations must be ultimate, and that at least one such ultimate asymmetrical relation must be a component in any asymmetrical relation that may be suggested".* He also claims that the argument shows that *"we are forced to admit what the theory was designed to avoid, a so-called 'external' relation, i.e. one implying no complexity in either of the related terms"* (§214).

Now it is one thing to hold that relations are *"ultimated"*, and quite another to hold that they are *"external"* in Russell's sense. Russell it seems conflates the claims, no doubt because it was Leibniz' strategy to argue from the fact that relations inhered in subjects to the conclusion that they are not ultimate. But suppose we grant what seems unavoidable -that relations are ultimate, that they constitute a distinct and ultimate category of existence and cannot be construed as quantities or qualities. This seems to be all that is established by the preceding argument. What we are still lacking is an argument that a relation exists independently, as a distinct thing -that relations are external. When Russell says that *"an adjective involving reference to M is plainly an adjective which is relative to M, and this is merely a cumbersome way of describing a relation"*, one can agree with him without agreeing that relations are therefore independent existences. When Russell points out that *"A is greater than B"* presupposes the relational proposition that *"The magnitude of A is greater than the magnitude of B"*, one can agree, while maintaining that the relation mentioned in the latter proposition inheres in A as much as that mentioned in the former proposition.

Russell's metaphysical views in **Principles** were greatly influenced by G.E. Moore and constitutes a revival of Platonism<sup>9</sup>. A very clear statement of his platonism is the following: *"The so-called prop-*

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<sup>9</sup> The view has been appropriately dubbed "Platonic atomism" by Peter Hylton, *Russell, Idealism and the Emergence of Analytic Philosophy*, Clarendon Press, Oxford, 1990, pp. 103-275.

*erties of a term are, in fact, only other terms to which it stands in some relation; and a common property of two terms is a term to which both stand in the same relation" (§216). For example, the proposition "Socrates is human", which seems to assert that Socrates has the property *humanity*, should be understood, Russell says, as the assertion that one term, *Socrates*, stands in a certain relation to another term, *humanity*. Likewise, to say that both Socrates and Aristotle are human is to say that each bears a certain relation to the term *humanity*. Presumably the same analysis is to be applied to relations, which is apparently what it means to say that they are external. Thus, to say that **L is greater than M** would be to say that **L** and **M** are each related to some third thing, distinct and independent from them -the relation *greater than*.*

There are many problems which arise concerning this sort of view; I shall consider only three, which have to do with (1) the unity of a proposition; (2) a vicious regress of relations which seems to be a consequence, and (3) accounting for the order of related terms.

1) **The problem of the unity of a proposition.** If a relation is a distinct and independent entity, then why is a relational proposition not simply a list of three things? Consider the proposition "**L is greater than M**". This proposition contains three terms: **L**, **M**, and *greater than*. Now the proposition is clearly not the same as the list, since the order of terms is apparently indifferent in the list, but not in the proposition. Furthermore, the list is not capable of being true or false, as is the proposition. Additionally, the proposition mentions one thing (**L's being greater than M**), but the list mentions three things. But then what is it that makes the proposition different from the list?

Russell's explanation is that, in the proposition but not in the list, *greater than* occurs as a something "*said of*" **L** and **M**; that is, *greater than* occurs in the proposition other than as a subject. But this answer

simply puts off the problem, for clearly now the question becomes: How can *greater than* occur other than as a subject, if it is an independent existent? Russell in fact recognizes this problem. In various passages in **Principles**, he distinguishes between, "*relation in itself*" and a "*relation as relating*". He says that, in a proposition such as "*L is greater than M*", the relation *greater than* occurs "*as relating*"; whereas, in a list of the constituents of a proposition such as "*L, M, greater than*", the relation *greater than* occurs "*in itself*" and not "*as relating*".

Russell incorrectly thinks that the need for this distinction derives from the nature of analysis, whereas in fact it derives from his view of relations:

*"A proposition, in fact, is essentially a unity, and when analysis has destroyed the unity, no enumeration of constituents will restore the proposition. The verb, when used as a verb, embodies the unity of the proposition, and is thus indistinguishable from the verb considered as a term, though I do not know how to give a clear account of the precise nature of the distinction (§54, cp. §99)"<sup>10</sup>.*

But the problem for Russell is not that of "*giving an account of the nature of the distinction*", but rather that of explaining how the distinction can exist at all. The "*as*" (or *qua*) idiom is typically used for speaking about different aspects of the same object. So in speaking about a relation *qua* relating, or *quo* existing in itself, one is admitting that one and the same thing -the relation- has, in different contexts, different aspects and thus different properties. But to admit this is to admit a distinction between subject and accident in the case of relations. (It would of course be absurd for Russell to say, consis-

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<sup>10</sup> In this passage, Russell speaks of relations as verbs, since he supposes that "all verbs are ... relations" (§54).

tently with his theory, that a relation "*as relating*" is simply one that is appropriately related to the term *relating*, and a relation "*in itself*" is one that is related to the term *in-itselfness*.)

2) **The problem of a vicious regress in relations.** If it is the case that, when two things are related, then there is some third thing, a relation, to which they are related, then there must be relations which relate the terms of a relation to the relation, and relations relating those relations to the relation, and so on. If one tries to understand the proposition "*L is greater than M*" by explaining it as the proposition that "*L is related to **greater than** which is related to M*", then one has introduced two additional entities -the relation of **L** to *greater than*, and the relation of *greater than* to **M** -in order to explain the original proposition. But these entities must themselves be explained in a similar manner, which initiates an infinite regress.

Russell is aware of this problem; it is in fact very similar to one raised against the very notion of a relation by F.H. Bradley in **Appearance and Reality**, chapter III. Russell replies to it by distinguishing two types of vicious regress: [i] a regress in the meaning of a proposition, and [ii] a regress in the implications of a proposition. Russell thinks, surely correctly, that a regress of type [ii] is unproblematic. For example, the proposition: "O is a number, and the result of adding 1 to a number is a number" implies an infinity of propositions: "*O is a number*"; "*1 is a number*" "*2 is a number*"; and so on. There is nothing vicious about such a regress; in fact, the power of human thought seems to consist in its being able to grasp a virtual infinity of propositions, by actually grasping a finite number of propositions which imply them. But a regress of type [i] is problematic; for it implies that the proposition asserted has no definite meaning, since its meaning is essentially incomplete. Furthermore, since a proposition would then contain an infinity of propositions, it is doubtful whether it could be grasped or asserted at all.



The question, then, is over what type of regress occurs in a relational proposition, if one adopts Russell's view of relations as external. Russell himself nicely states the relevant considerations:

*"It may be urged that it is part of the very meaning of a relational proposition that the relation involved should have to the terms the relation expressed in saying that it relates them, and that this is what makes the distinction, which we formerly (§54) left unexplained, between a relating relation and a relation in itself. It may be urged, however, against this view, that the assertion of a relation between the relation and the terms, though implied, is no part of the original proposition, and that a relation is distinguished from a relation in itself by the indefinable element of assertion which distinguishes a proposition from a concept" (§99).*

He concludes, rather lamely, that *"it seems impossible to prove that the endless regress involved is of the objectionable kind"*.

However, it does seem that the regress is of the vicious sort. First of all, Russell's alternative suggestion can be put aside. An assertion is an action, but a proposition, which is what is asserted, presumably is not. Even supposing that there is an *"indefinable element of assertion"*, which has to do with the unity of an act of assertion, it cannot be correct to hold that this accounts for the unity of *what* is asserted. To hold this would be to hold that the unity of a proposition is merely imposed by the person who asserts it, and hence it becomes *"ideal"*, which Russell would not wish to accept. Yet if the *"element of assertion"* which gives unity to the proposition is not something contributed by the speaker, it must be in the proposition itself. However, according to Russell, the analysis of a proposition fails to reveal any such element. Hence this attempt to explain the unity of the proposition is unacceptable.

Moreover, Russell's remark that "*the assertion of a relation between the relation and the terms, though implied, is no part of the original proposition*" misses the point of the objection. Of course it is not the case that someone who asserts, for example, "*L is greater than M*" also asserts "*L is related to greater than, and greater than is related to M, and L is related to related to greater than, etc., etc.*" -it is precisely the point of this objection that this infinity of claims could *not* be asserted in a single proposition. The objection contends, rather, that they would *have to be* asserted, given Russell's account of relations, if what is meant in asserting the proposition "*L is greater than M*" is to have a unity -and Russell says nothing in response to this point.

3) **The problem of order in relations.** Russell seems unaware of this difficulty, unlike the first two. As we have seen, Russell holds that for two terms to be related is simply for them to be related to some third thing, a relation. The difficulty then arises: What makes a relation between two things distinct from a property common to them? We quoted Russell as saying that "*a common property of two terms is a term to which both stand in the same relation*". One might wonder why it is necessary that the two terms stand to this term in the same relation. For it would seem that that which explains why the two things have one and the same property is the mere fact that both are related to one and the same term, not that both have the same relation to that term. Suppose, for example, that there are two yellow things, which therefore have something in common, viz. *yellow*. That they have something in common is to be explained as: there is some one thing, *yellow*, to which the two things are related. The fact that, in addition, their relation to *yellow* is exactly the same seems unimportant and unnecessary to the explanation.

Now if the relations of two yellow things to *yellow* were indeed different, we would not therefore say that *yellow* has, as a consequence,

become a relation between those things, and is no longer a property. Rather, two things differently related to yellow would be simply two things that are *yellow* or have yellowness in different ways. It follows, then, that the mere fact that two things are related to the same concept by different relations, does not suffice to explain why it is that that concept is to be counted as a relation between the two things rather than as a property common to them. What one needs in addition, in order to have a relation, is a concept which somehow proceeds from one term to the other; that is, a concept which has a "*sense*". This is not surprising, for, as we saw and as Russell agrees, it is essential to a relation that it have a sense. However, if a relation is external to and independent from the terms related, it is difficult to see how it could have a sense, or proceed from one term to the other. An external relation, apart from its terms, is a single term, without distinction, and would not seem to admit of "*from*" and "*towards*". But if a term's having a sense and thus being a relation depends upon its having terms, then a relation is dependent on its terms and in that regard at least is accidental to them. Hence the doctrine of relations as external seems unable to explain why order is important in a relational proposition, since it is unable to explain how relations can have a sense"<sup>11</sup>.

The reason why Russell overlooked the problem of order perhaps derives from his manner of typographically representing a relation. He represents "*a bears the relation **R** to b*" as "**aRb**", in which case the variable representing the referent, **a**, bears a distinctive, asymmetrical spatial relation to the variable representing the relation, **R**, which

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<sup>11</sup> Russell's very manner of explaining sense seems to rely on something other than relations understood as external. To say that a relation "proceeds from one thing to another", is clearly not to say that there is another relation (say, of procession) to which that relation is related. It is, rather, to attribute to the relation an internal structure which is complex and relational.

in turn bears a distinctive, asymmetrical spatial relation to the variable representing the relatum, **b**. The symbols have a sense, because spatial relations have a sense. Russell might then overlook this, and think that **a**'s having any relation to **R**, and **R**'s having any relation to **b**, would still preserve the sense of the relational claim. The difficulty about order would be clearer if a relation were depicted by having **a** and **b** below **R**, in any order, and merely having lines drawn from **a** and **b** to **R**. Then it would be evident that the relation's having sense would have to be a consequence of those lines' representing the right sort of "sense-conferring" relations -which seems impossible.

### 3. Topics in the Theory of Relations

I shall consider three questions discussed at length by Russell and which remain of interest today: (1) Can relations be understood extensionally?, (2) Are there reflexive relations?, and (3) Can there be simple relations which relate more than two terms?

The predominant view of **Principles** is that relations exist intensionally as well as extensionally. It should be said that this is *prima facie* a difficult view for Russell to maintain. We saw that Russell's dislike for any sort of idealism compels him to assign no role for the activity of mind in the classification or organization of knowledge; he understands logic to be a very general science of the world. He understands the mind's contact with reality to be direct and unmediated by concepts or ideas. Consequently, it is difficult to see how he can allow for intensionality at all. He attempts to explain intensionality with regard to classes by means of his peculiar theory of denoting concepts (**Principles**, chapter V). Yet he has no similar means of explaining relations in intension.

In §98 of **Principles**, Russell gives four arguments against the reduction of relations to extensions:

(i) Since an extension is simply a class, to consider a relation as an extension would be to consider it as a class of two-member classes. However, the order of the elements of a class is inessential to the identity of a class, since two classes are identical just in case they have the same members. Yet the order of the elements of a relation is, as we have seen, essential to a relation. Hence, a relation cannot be regarded as a class of two-member classes -for to do so would destroy the order of the related terms, and various unwanted consequences would immediately follow. For example, the relation "*greater than*" would be the same as the relation "*less than*"; for the former would be the class of all two-member classes whose elements are related as greater to lesser, and the latter would be the class of all two-member classes whose elements are related as lesser to greater; but since a relation always implies its converse, corresponding to each class in the former there would be exactly one class in the latter, and vice versa; hence, the classes would be equivalent; and if the classes, then also the relations, which is absurd.

(ii) To propose a definition of a relation which captured the notion of a 'sense', it would be necessary to make use of a relation having a sense; thus sense cannot be explained away and must be taken as fundamental; but a relation in extension does not have a sense; thus relations must also exist in intension.

(iii) In order to define any sort of thing in terms of some other sort of thing, it is necessary first to give a general characterization of the former sort of thing. But any general characterization of relations would have to rely upon the (unreduced) notion of a relation -for example, to begin to talk about the relational proposition "*a bears R to b*" it would be necessary to say something like "*a is the referent, b the relatum*", but this assertion would presuppose some relations, e.g. the relation of *a* to existence, the difference between *a* and *b*, etc. But such relations have sense and exist in intension; and thus it is impossible to dispense with relations in intension.

(iv) Propositions can be constructed about relations, e.g. "*Different is different from same*". But one cannot speak about what does not exist. Thus these relations must exist, and they are evidently distinct from the terms which are related through these relations. Thus, relations do not exist only in extension.

Of these four arguments, (i) and (iv) can be put aside immediately. Argument (iv) depends upon the Parmenidean view that "*what is not cannot be thought or uttered*". Russell is obliged to adopt this principle, since he will not countenance ideas or concepts in the mind of a thinker; but we have no reason to adopt it. Argument (i) can be dismissed because it is now recognized that an ordered pair can be defined in terms of classes, in such a way as to preserve the order of the elements. All that is necessary is that, given the class meant to represent an ordered pair, it is possible to determine which element comes first in the ordered pair which it represents. There are two common conventions for this; an ordered pair  $\langle \mathbf{a}, \mathbf{b} \rangle$  can be defined either as:

$$\begin{aligned} & \{\{\mathbf{a}\}, \{\mathbf{a}, \mathbf{b}\}\} \\ \text{or } & \{\{\mathbf{a}, \mathbf{a}\}, \{\mathbf{a}, \mathbf{b}\}\}. \end{aligned}$$

According to the first convention, the element of the single ton set is taken to be the first member of the pair; according to the second, the element repeated in a class is treated as the first member.

Arguments (iii) and (iv), however, are serious arguments, which concern the practical impossibility of dispensing with relations regarded intensionally. Both arguments can in fact be applied to the definitions of an ordered pair given above. Note that, as urged in (ii), the above definitions themselves make use of relations not in extension: both rely on spatial relations to depict the inclusion relation 'E' which is fundamental to set theory, and the inclusion relation itself,

which is used in the definitions, cannot without circularity be defined in extension. Furthermore, as urged in (iii), that which is to be defined involves a relation not in extension, namely, the ordering peculiar to the ordered pair, which is not captured or replaced so much as encoded or represented by the set-theoretic definition.

I turn now to the second question, whether there are reflexive relations. Russell says that "*there is a certain temptation to affirm that no term can be related to itself; and there is a still stronger temptation to affirm that, if a term can be related to itself, the relation must be symmetrical, i.e. identical with its converse*" (§95). Russell does not say what the temptation is, but it is easy enough to guess: There can be a relation between two terms, only if there are two terms; but if there is no distinction between the terms, then they cannot be two; hence, if there is no distinction between the terms, there cannot be a relation between them. But a single thing, *qua* single, is not distinct from itself; hence, a single thing *qua* single can have no relation to itself. Indeed, it could have a relation to itself only if it were somehow regarded as distinct from itself. But regarding it so does not make it so -it would in fact be false so to regard it. Hence nothing actually has a relation from itself to itself; and thus there are no reflexive relations. Russell replies to this temptation as follows:

*"if no term were related to itself, we should never be able to assert self identity, since this is plainly a relation. But since there is such a notion as identity, and since it seems undeniable that every term is identical with itself, we must allow that a term may be related to itself"* (§95).

That is, identity is a relation, and each thing is identical with itself; thus each thing has at least one reflexive relation; therefore, reflexive relations exist.

The argument seems a good one, but is Russell entitled to advance it, and what exactly does it establish? At first glance, one would think that Russell would be more strongly tempted than he is by the argument against reflexive relations. If a relation somehow exists *between* the terms related by it, then the relation of identity, if it were reflexive, would exist *between a thing and itself*, which seems impossible.

Moreover, it is a consequence of Russell's view of relations, that there are in fact no reflexive relations after all. For if relations are external, then the relation of identity is external to the thing related by it. Thus, to say that something is identical to itself is just to say that there is a distinct thing, the relation *sameness*, to which that thing is related, and which bears a relation to that thing - in which case there would be one relation, relating a thing to the relation *sameness*, and another and different relation, relating the relation *sameness* to that thing. Both relations would be necessary, for the relation to be reflexive. But clearly on this way of understanding self-identity, there is no true reflexivity at all, because there are no ultimate reflexive relations: on the contrary, this theory construes self identity as a reciprocal pair of non-reflexive relations relating a thing to something else, to *sameness*.

Furthermore, there is a kind of vicious, ontological regress, an infinite multiplication of entities, which results from admitting self identity, if that relation is understood as an external relation, and if relations are also terms, as Russell claims. For suppose that each thing is identical with itself. Then for each thing there is a pair of relations reciprocally relating it to the relation *sameness*. But these relations are terms which are themselves self-identical, and hence they also are reciprocally related to the relation *sameness* by other relations, which are themselves terms, and thus self-identical, etc. Hence, from the seemingly barren tautology that everything is identical with itself, one is able to generate an infinity of self-identical entities.



Clearly, in §95 Russell is not carefully considering the implications of admitting self identity. The principle that everything is self-identical is needed for the logicist program, and, again, not distinguishing logic and metaphysics, he therefore draws an ontological conclusion about reflexive relations. Yet that conclusion cannot be supported by the metaphysical theses about relations which he has defended. In another section of **Principles** Russell in fact expresses doubts about the existence of any reflexive relation: *"such relations, in any system, constitute a grave logical difficulty; ... they would, if possible, be denied philosophic validity; and... even where the relation asserted is identity, there must be two identical terms, which are therefore not quite identical"* (§198). Russell then adds *"As this raises a fundamental difficulty, which we cannot discuss here, it will be prudent to allow the answer to pass"* -and he then refers the reader in a footnote to §95, the passage we have been discussing. But in that passage, as we saw, Russell simply asserts but in no way defends the existence of reflexive relations.

The third and final question concerns the existence of relations having more than two terms. Russell holds that there can be relations with any number of terms. It is difficult to see how this is consistent with his claim that the notion of sense is essential to a relation. Consider, for example, a relation of three terms: **R (a,b,c)**. How is this to be read, in a way that exhibits the sense of the relation? If we say, for example, that *"a bears R to b and c"*, then we seem to treat **b** and **c** as a unity, and we say that **a** has the relation **R** to this unity<sup>12</sup>. But if we do not treat two of the terms of the relation as a unity, we must somehow distinguish them -and then we must also relate them, if the proposition is to have a unit. However, if we do this,

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<sup>12</sup> And likewise if we say "a and b bear R to a," or "a and c bear R to b", or any of the other possible variations.

it seems that **R** is no longer a simple relation, but rather consists of a complex of two-term relations, e.g. "*a bears R to b bearing R1 to c*". In short, it seems that to deny that relations must have two terms requires abandoning the view that a relation has a sense which determines a certain order among its terms.

Russell gives three arguments against the view that relations must have two terms: (i). There are no a priori reasons, he asserts, for restricting relations to two terms. (ii) There seem to be examples of relations with more than two terms, such as: asserting that a collection of  $n > 2$  members has **n** members; or asserting that one thing is between two others. (iii) Propositions about a thing's location in space and time, he says, can be reduced to those involving two-term relations only "*by a very cumbersome method*".

These arguments vary greatly in character. (i) is an attempt to shift the burden of proof; yet it is weak, since, as we saw, there are indeed a priori reasons (taking that phrase loosely) for the view that relations must have two terms -reasons which concern the fact that a relation essentially has a sense.

Neither of the examples mentioned in (ii) requires three-term relations. Predications of number always seem to presuppose some grouping of the things numbered: "*There are twelve months*", or "*There are seven continents*". In order to count something, it is necessary to know what kind of thing one is counting; even a very vague assertion involving number such as "*This and that are two*" involves treating the things indicated as members of the same kind -this and that are the *things I before me, or the things I have pointed out*. These points have of course been argued for at length and established by Frege in his **Foundations of Arithmetic**<sup>13</sup>. Frege understood the predication of a number to group to be the assignment of a number

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<sup>13</sup> FREGE, G.: *Foundations of Arithmetic*, 2nd. ed., trans. J. L. Austin, Basil Blackwell, Oxford, 1953.

(which he took to be an object) to a concept. If one adopts that view, then the predication of a number involves only a two-term relation. But whether a number is an object or a property, the important thing to notice is that things which are numbered must have a unity, so that the number predicate is applied to a group having a unity. Hence, if a predication of number involves a relation at all, it involves a relation between two terms, the group which is to be counted and the number<sup>14</sup>

The relation between is evidently not an irreducible three-term relation, since it is clear that to say that  $y$  is between  $x$  and  $z$  is just to say that there is some asymmetrical and transitive, two-term relation  $R$ , such that  $xRy$  and  $yRz$ . Russell in fact concludes a rather lengthy discussion of between in **Principles** §§197-202 by saying that propositions involving between are appropriately analyzed into two-term relations: he says that the definition just mentioned "*gives not merely a criterion, but the very meaning of betweenness*".

With regard to (iii), Russell gives an odd argument for the thesis that existence at a time is an ultimate three-term relation:

*"If 'A exists now' can be analyzed into 'A is now' and 'A exists', where exists is used without any tense, we shall have to hold that 'A is then' is logically possible even if A did not exist then; for if occupation of a time be separable from existence, a term may occupy a time at which it does not exist, even if there are other times when it*

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<sup>14</sup> How it is that a group of distinct things can be united is a difficult question. I suppose it could be argued that, when a whole is "greater than the sum of its parts", it is necessary that each part of the whole be related to every other part of the whole. Russell does not consider such cases as potential examples of relations with more than two terms. In fact, it is difficult to see how he can admit that a whole can be "greater than the sum of its parts" his view that relations are external would seem to exclude this ever being possible.

*does exist. But, on the theory in question, 'A is then' and 'A exists' constitute the very meaning of 'A existed then', and therefore, when these two propositions are true, A must have existed then. This can only be avoided by denying the possibility of analyzing 'A exists now' into a combination of two-term relations ... " (§444).*

It can be seen that the argument is easily extended to any sort of predication. For example, suppose "*A is infact yellow*" is to be analyzed as <"*A exists*" & "*A has yellow*" >. But it is logically possible that *A* be red rather than yellow; hence we can formulate the proposition "*A has red*". Yet we already have available to us "*A exists*", hence we can combine these and get <"*A exists*" & "*A has red*" >, which is by hypothesis equivalent to "*A is red*". It follows that "*A is yellow*" and also "*A is red*". And by a similar argument we could show that *A* is anything whatsoever.

But it is clear that Russell's argument has force only if one distinguishes, as Russell does, between the *is* of predication and the *is* of existence. In **Principles** Russell draws a distinction between what he calls "*existence*" and "*being*" (cf §§ 47,427). Existence is actual existence; being is the sort of existence possessed by mathematical objects and mere objects of thought. Russell's view is that, by stet a predicate to a subject in thought, one is granting "*being*", though not actual existence, to that combination of subject and predicate. Now if it is possible in this way to predicate something of a subject without predicating actual existence, it is clear that, in order to assert that a subject actually has a predicate, something more is needed than the assignment of a predicate to a subject. It becomes necessary, in fact, to assign a specific predicate, the predicate "*existence*", to the subject, over and above whatever else is assigned to the subject. But once this is done for any predicate at all, the subject has "*existence*", and then, by the argument above, one is able to confer actual existence to any predicate at all which may be assigned to the subject in thought.

Russell did not recognize that this argument about existence at a particular time could be extended to any sort of predication, and that the argument depends crucially on the assumption that a distinction can be drawn between *being* and *existence*, (and thus between the *is* of predication and the *is* of actual existence). Russell wished to maintain that distinction, but to avoid the unwanted conclusions of the above argument, by treating predications of spatial and temporal location as a special case, involving ultimate three-term relations. However, since the argument in fact concerns any sort of predication, it obliges us either to regard all predications of existence as involving ultimate three-term relations, or to reject the distinction between existence and being which gives rise to the unwanted consequences. The latter alternative seems no less plausible than the former; hence (iii) does not provide us with any compelling reason for regarding some predications as involving ultimate three-term relations.

With regard to the question of whether there are any relations with more than two terms, in sum: Russell provides no arguments at all against a negative answer to the question, and none of his arguments for an affirmative answer is compelling.

#### 4. Conclusion

Russell's **Principles of Mathematics** adopts a bold platonic theory of the objects of mathematics, of logic, and of thought. The fundamental theses of the book are ultimately incoherent and quickly lead to absurd consequences, as did Platonism originally. Nonetheless, it is precisely his platonism which leads Russell to take logical and metaphysical difficulties seriously, and to seek dialectically for determinate solutions to them. If we cannot embrace his conclusions, we can, however, admire the spirit of his inquiry; and we can in addition

regret the fact that analytic philosophy, through Wittgenstein and logical positivism, would later aim to reject rather than correct that tradition's original metaphysical aspirations. Aristotle remarks in his **Metaphysics** that it is necessary to become acquainted with the difficulties surrounding a metaphysical question in order to be able to find and recognize the proper solution. If this is the case, then Russell's **Principles of Mathematics**, so rich in dialectical argument and metaphysical puzzles, is a useful book indeed. The above discussion of its arguments concerning relations, however brief, provides, I think, a partial confirmation of this conclusion.